Lecture 5

New topic **thermal history** (next few lectures):

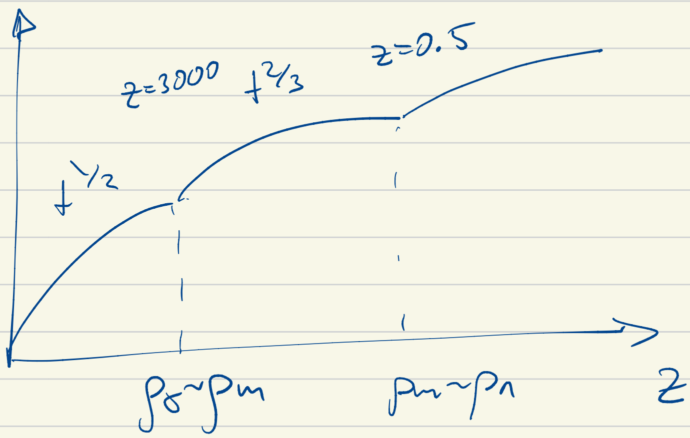
- Thermal history, evolution of matter and radiation, CMB, BBN (briefly Baryogenesis)

This lecture:

- Review (distances and FRW)
- Equilibrium statistical physics
- Temperature of CMB
- Boltzmann equation

• Review

Epochs of the Universe:



$$x = \frac{a}{a_0}, \quad A^2(x) = \Omega_\Lambda + \Omega_k \frac{1}{x^2} + \Omega_m \frac{1}{x^3} + \Omega_\gamma \frac{1}{x^4}$$

Luminosity distance:

$$d(z) = (1+z)\,\frac{1}{H_0}\int_{\frac{1}{1+z}}^{1} \frac{dx}{x^2 A(x)} \qquad [k=0 \text{ case}]$$

[sepheids]

Hubble law:

$$\frac{dl_p}{dt} = \dot{a}\, l_c = \frac{\dot{a}}{a}\, l_p = H l_p$$

redshift:

$$z(t) = \frac{\lambda_0 - \lambda_1}{\lambda_0} = \frac{a(t_0)}{a(t_1)} - 1$$

- Equilibrium statistical physics

The goal of the next few lectures is to understand how various matter components of the universe (photons, electrons, protons, neutrons, neutrinos and some light atoms) exchange energy and come in and out of thermal equilibrium.

Full description is given by the density matrix:

$$\hat{\rho} = \frac{1}{Z} \exp\left(-\frac{\hat{H}}{T} + \mu_i \hat{Q}_i \right)$$

chemical potentials

conserved numbers

In many cases, once in equilibrium, we can describe components as free particles, described by distribution in momentum space.

• Note that really free particles never thermalize

$$N = \frac{g}{(2\pi)^3} \int d^3p \, n(p)$$

$$\rho = \frac{g}{(2\pi)^3} \int d^3p \, E(p) \, n(p)$$

$g \rightarrow$ number of degrees of freedom.

$n(p) \rightarrow$ thermal distribution function

$$n_{B/F} = \frac{1}{\exp\left(\frac{E(p) - \mu}{T}\right) \mp 1}$$

$-$ → Bosons

$+$ → fermions

$$E(p) = \sqrt{p^2 + m^2}$$

In the relativistic limit we can compute N and ρ analytically.

$T \gg m_i \Rightarrow E \sim |p| \sim T$

$$\rho_i = \frac{g_i}{(2\pi)^3} \int d^3p\, |p| \cdot \frac{1}{e^{\frac{|p|}{T}} \pm 1} =$$

$$= \frac{g_i}{2\pi^2} \int dE \frac{E^3}{e^{\frac{E}{T}} \pm 1} = \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{Bose} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{Fermi} \end{cases}$$

For N one gets
$$\begin{matrix} \frac{\zeta(3)}{\pi^2} g_i T^3 & B \\ \frac{3}{4} \frac{\zeta(3)}{\pi^2} g_i T^3 & F \end{matrix}$$

$\zeta(x)$ is the Rieman Zeta-function

$\zeta(3) \simeq 1.2$

energy per particle: $\langle E \rangle = \frac{\rho_i}{N_i} \simeq \begin{cases} 2.7\, T & B \\ 3.15\, T & F \end{cases}$

It is common to define the "effective number of species"

$$g_* = \sum_{\text{bosons}} + \frac{7}{8} \sum_{\text{fermions}} \; ;$$

$$\rho = \frac{\pi^2}{30} g_* T^4$$

• Entropy

For $\mu = 0$ we have

$$U = TS - pV$$

For $V \to$ comoving (not physical)

$$s = \frac{S}{V} = \frac{\rho + P}{T} = \frac{4}{3} \frac{\rho}{T}$$

$$p = \rho/3$$

- Non-relativistic limit

$$m \gg T \quad , \quad m - \mu \gg T \quad , \quad E = m_i + \frac{p^2}{2m_i}$$

(μ can be important if non-relativistic)

$$N_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{\frac{\mu_i - m_i}{T}}$$

[derive - ?]

$$\rho_i = g_i m_i N_i$$

$$P_i \simeq g_i N_i T \ll \rho$$

• CMB Temperature

CMB played (and is playing) a very important role in development of cosmology (see Colloquium by Peebles). We will study it in details, but now we just want to estimate its temperature.

If radiation dominates the universe

$$H = \left(\frac{8\pi G}{3}\rho_{rad}\right)^{1/2} = \frac{T^2}{M_0}$$

$$\rho_{rad} = \frac{\pi^2}{30} g_* T^4$$

$$M_0 = \left(\frac{45}{4\pi^3 g_* G}\right)^{1/2} = \frac{M_{pl}}{1.66\, g_*^{1/2}}$$

where g_* is the number of species that dominate in the radiation epoch.

$$a = a_0\left(\frac{t}{t_0}\right)^{1/2} \Rightarrow H = \frac{\dot a}{a} = \frac{1}{2t} \Rightarrow$$

$$\frac{1}{2t} = \frac{T_{rad}^2}{M_0}$$

This equation is rather precise at T_{eq} when $\rho_m \sim \rho_\gamma$ (and earlier)

$$T_{eq} \sim 1eV, \quad t_{eq} \sim 7\cdot 10^5\, y$$

If we put current time $t \sim 10^{10}\, y$ we get $T_\gamma \sim 10K$ (instead of 2.5K which is the real value)

This is because after t_{eq}.

$$T_\gamma \sim \frac{1}{a}, \quad \text{but} \quad a \sim t^{2/3}, \text{ not } t^{1/2}$$

but $\frac{t^{2/3}}{t^{1/2}} \sim t^{1/6}$ and $z_{eq} \sim 10^3$

$(10^3)^{\frac{1}{6}}$ is not large!

This is in the spirit of early days estimates in Big bang theory. Now we can do much better, but being able to get order of magnitude estimates easily is very important!

g_* in the full standard model:

Scalar:

Higgs : $1 + 3 \rightarrow$ Goldstones

Fermions:

quarks: u, d, c, s, t, b (L and R) $\times$

$\times$ 3 colors + anti-quarks $\rightarrow 72$

Leptons:

e_L, e_R, ν^e_L No $SU(3)$ +

μ_L, μ_R, ν^μ_L + anti-leptons

$\tau_L, \tau_R, \nu^\tau_L$ $\rightarrow 18$

Vectors:

photon + gluons $= 9 \times 2$

W^+, W^-, Z $\to$ massive $= 3 \times 3$

Total:

$$1 + 9 \cdot 2 + 3 \cdot 3 + \frac{7}{8}(72 + 18) = 106.75$$

[at energies $E \gg m_{ew} \sim 150$ GeV]

- We do not know if the universe was ever that hot, but if it was, it would have $g_* \sim 100$

- Particle kinetics (Boltzmann equation)

We now study how a gas of particles approaches equilibrium in an expanding universe. That is, we first assume a generic distribution function (isotropic and homogeneous)

$$n(p, t)$$

$\vec{p} \to$ physical momentum. In the absence of interactions comoving momentum is conserved:

$$x_p = a x_c \Rightarrow p_p = \frac{p_c}{a}$$

$$n(p,t) = n_0\left(p \frac{a}{a_0}\right)$$

Let's derive the infinitesemal change:

$$\frac{\partial n}{\partial t} = n' \cdot p \cdot \frac{\dot{a}}{a_0}$$

$$\frac{\partial n}{\partial p} = n' \cdot \frac{a}{a_0}$$

$$\frac{\partial n}{\partial t} - Hp \frac{\partial n}{\partial p} = 0$$

If we integrate over p we get

$$\frac{dN}{dt} - \int \frac{d^3 p}{(2\pi)^3} Hp \frac{\partial N}{\partial p} = 0$$

$$\left\lceil \int_0^\infty dp \cdot p^3 \frac{\partial f}{\partial p} = 3 \int_0^\infty dp \cdot p^2 f = 3 \int d^3 p \, f \right\rfloor$$

$$\frac{dN}{dt} + 3HN = 0$$

This was for non-interacting particles.
Note that they don't stay in a thermal state (generically)